THE MCKEAN-SINGER FORMULA VIA EQUIVARIANT QUANTIZATION

EUGENE RABINOVICH

1. Introduction

The aim of this talk is to present a proof, using the language of factorization algebras, and in particular the index theorem in Chapter 7 of [G], of the following

Theorem 1.1 (McKean-Singer). *Let V be a Hermitian, $\mathbb{Z}_2$-graded vector bundle on a compact Riemannian manifold M, with $|dx|$ the Riemannian volume form on M. Let D be a self-adjoint Dirac operator on V, with k_t the heat kernel of D^2. Then*

$$\operatorname{ind}(D) = \int_M \operatorname{Str}(k_t(x,x))|dx|. \tag{1}$$

The actual McKean-Singer theorem works for non-self-adjoint Dirac operators as well, but our proof will require D to be self-adjoint. We will give definitions of all of the objects in the theorem shortly, but first a bit of philosophy. This theorem gives us a relationship between a global, analytic quantity (the index of a Dirac operator) and a local, physical quantity (the super-trace of a heat kernel). This is what the index theorem is most famous for. We will see that the theorem of Gwilliam is similar in nature: it describes two ways to compute the obstruction to quantizing a field theory equivariantly with respect to the action of an L_∞ algebra. One involves Feynman diagrams (which involve heat kernels), and the other is a global characterization (which will give us the index). This is, very roughly speaking, why we are able to use the theorem relating to field theory to prove an index-type theorem.

2. Generalized Laplacians, Heat Kernels, and Dirac Operators

We present here a list of definitions and results relevant to our talk. Throughout, M is a Riemannian manifold with Riemannian volume form $|dx|$. We let $V \to M$ be a vector bundle, which we will eventually specialify to be $\mathbb{Z}_2$-graded. We let $\mathcal{V}$ be the sheaf of smooth sections of V.

Definition 2.1. A **generalized Laplacian** is a differential operator

$$H : \mathcal{V}(M) \to \mathcal{V}(M)$$

such that

$$[[H,f],f] = -2|df|^2,$$

where we are thinking of C^∞ functions as operators corresponding to multiplication by those functions.

Now we let V be $\mathbb{Z}_2$-graded, and we denote by $V^\pm$ the plus or minus graded components of V.

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Definition 2.2. A **Dirac operator** on V is a grading-reversing operator

$$D : \Gamma(M, V^{\pm}) \to \Gamma(M, V^{\mp})$$

such that D^2 is a generalized Laplacian.

Theorem 2.1 (The Heat Kernel)**.** *Let (E, Q) be an elliptic complex, and let Q^{GF} be a cohomological degree -1 operator such that $H := QQ^{GF} + Q^{GF}Q$ is a generalized Laplacian. Then there is a unique* ***heat kernel*** $k \in \Gamma(M \times M \times \mathbb{R} > 0, E \boxtimes E^*)$ *satisfying:*

(1)
$$\frac{d}{dt}K_t + (H \otimes 1)K_t = 0$$

(2) *For $s \in \Gamma(M, E)$,*
$$\lim_{t\to 0} \int_{y\in M} k_t(x, y)s(y)|dx| = s(x),$$
where the limit is uniform over M and is taken with respect to some norm on E.

The heat kernel is the kernel of the operator e^{-tH} in the sense that

$$\int_{y\in M} k_t(x, y)s(y) = (e^{-tH}s)(x).$$

Definition 2.3. Let D^+ denote the restriction of a self-adjoint Dirac operator D to the space of positively-graded sections. Then, the **index** $\mathrm{ind}(D)$ of D is $\dim(\ker(D^+)) - \dim(\mathrm{coker}(D^+))$.

The last definition we need to understand this theorem as stated is

Definition 2.4. If $\phi : V \to V$ is a grading-preserving endomorphism of the super-vector space V, then the **supertrace** $\mathrm{Str}(\phi)$ is defined to be

$$\mathrm{Str}(\phi) = \mathrm{Tr}(\phi\mid_{V^+}) - \mathrm{Tr}(\phi\mid_{V^-})$$

3. Equivariant Quantization of Free Theories

In this section, we discuss a theorem of Gwilliam about the quantization of cotangent theories with the action of an elliptic local L_∞ algebra.

3.1. **Set-up.** For our purposes, we will just need the L_∞ algebra to have a 1-bracket. In other words, we will have an elliptic complex $(\mathcal{L}, d)$ on M acting on an elliptic complex $(\mathcal{E}, Q)$. Specifically, we need the following

Definition 3.1. A **local representation** of the elliptic complex $(\mathcal{L}, d)$ on an elliptic complex $(\mathcal{E}, Q)$ is a polydifferential operator $[\_, \_] : \mathcal{L} \otimes \mathcal{E} \to \mathcal{E}$ such that we have

(1) A derivation property:
$$Q([X, \phi]) = [dX, \phi] + (-1)^{|X|}[X, Q\phi]$$

(2) A Jacobi identity:
$$[X, [Y, \phi]] = (-1)^{|X||Y|}[Y, [X, \phi]]$$

Remark: As mentioned above, we could expand this definition to include local representations of elliptic L_∞-algebras on $(\mathcal{E}, Q)$, which is the level of generality in which the theorem of Gwilliam applies. However, we will not need this in our proof.

Example 3.1 (Key Example)**.** Let V be a $\mathbb{Z}_2$-graded vector bundle on M with a Dirac operator D. Define

$$\mathcal{E} = \mathcal{V}^+ \xrightarrow{D} \mathcal{V}^-.$$

Here $\mathcal{V}$ sits in degree 0. We let $\mathcal{L} = (\Omega^\bullet, d_{dR})$, and define for $f \in C^\infty M$

$$[f, \phi] = f\phi.$$

By the derivation property, we must have

$$[df, \phi] = D(f\phi) - f(D\phi).$$

Some thought shows that this gives a well-defined action of one-forms on $\mathcal{E}$. The brackets of all higher forms on elements of $\mathcal{E}$ vanish for degree reasons. Finally, the Jacobi property is trivially satisfied.

If we have the data of an elliptic complex $(\mathcal{L}, d)$ and a local representation $(\mathcal{E}, Q)$ of $\mathcal{L}$, we can define a BV theory whose space of fields is

$$\mathcal{F} := \mathcal{L}[1] \oplus \mathcal{E} \oplus \mathcal{E}^![-1],$$

where $\mathcal{E}^!$ is the space of sections of the bundle $E^\vee$. We think of this as the space of fields corresponding to the action

$$S(X, \phi, \psi) = \langle \phi, Q + Q^! \psi \rangle + \langle \phi, [X, \psi] \rangle$$

where $X \in \mathcal{L}[1], \psi, \phi \in \mathcal{T}^*[-1]\mathcal{E} := \mathcal{E} \oplus \mathcal{E}^![-1]$, and $\langle -, - \rangle$ is the natural anti-symmetric, degree -1 pairing on $T^*[-1]$. We should think of X as a background field, and we would like to quantize, for every X, the theory with action $S(X, \phi, \psi)$, thought of as only a function of the fields ϕ, ψ. The field X is non-propagating in the sense that when we do the Feynman diagrammatics, there are no internal edges corresponding to $\mathcal{L}$ fields; we think of X as being an external "source."

Another way to think about this setup is to think of X as providing a deformation of the complex $(\mathcal{E}, Q)$ with "differential" $Q + [X, -]$. This operator will be degree $+1$ if X lives in degree 1 in $\mathcal{L}$ and will square to zero if

$$Q^2\phi + Q[X, \phi] + [X, Q\phi] + [X, [X, \phi]] = [dX, \phi] = 0,$$

where we have used both properties of a local representation in the penultimate equality. Thus, for every closed degree 1 element X of $\mathcal{L}$, we have another elliptic complex $(\mathcal{E}, Q + [X, -])$.

Now is the right time to say something about the Feynman diagrammatic way to describe the situation. We should think of the term

$$\langle \phi, [X, \psi] \rangle$$

as corresponding to a trivalent vertex that we can put in graphs, with one half-edge corresponding to an element of $\mathcal{L}$ and two corresponding to $T^*[-1]\mathcal{E}$.

We will need one final bit of data to quantize the theory we're describing:

Definition 3.2. A **gauge-fixing** operator is an operator $Q^{GF} : T^*[-1]\mathcal{E} \to T^*[-1]\mathcal{E}$ satisfying

1. $(Q^{GF})^2 = 0$.
2. Q^{GF} is self-adjoint for the pairing $\langle -, - \rangle$.
3. $[Q, Q^{GF}]$ is a generalized Laplacian, which we will denote H. (Here, we are taking the graded commutator, which for degree $+1$ and -1 operators is the anti-commutator).

In our theory, we have an obvious choice of gauge-fixing operator, namely the Dirac operator $D^- + D^{-!}$.

3.2. **Chevalley-Eilenberg Cochains.** For those of you who are into factorization algebras, you know that given a space of fields we can define a cochain complex of classical observables for each open set of M. In our case, we can take

$$\mathrm{Obs}^{cl}(U) = C^\bullet\left(\mathcal{L}(U), \widehat{Sym}^*\left(\mathcal{E}^\vee \oplus \mathcal{E}^![1]^\vee\right)\right),$$

i.e. the classical observables are the Chevalley-Eilenberg cochains for the representation $\widehat{Sym}^*\left(\mathcal{E}^\vee \oplus \mathcal{E}^![-1]^\vee\right)$. Here $\vee$ means distributional dual. More explicitly,

$$\mathrm{Obs}^{cl}(U) = \widehat{Sym}^\bullet\left(\mathcal{L}[1]^\vee(U) \oplus \mathcal{E}^\vee(U) \oplus \mathcal{E}^![-1]^\vee(U)\right)$$

with the differential $d + Q + \{I, \_\}$. Here, d is the operator on $\mathrm{Obs}^{cl}(U)$ defined as the dual to d on $\mathcal{L}^\vee[-1]$ and 0 on $\mathcal{E}^\vee \oplus \mathcal{E}^![-1]^\vee$, and then extended to all of $\mathbf{Obs}^{cl}(U)$ by demanding that it be a derivation of degree 1. We mean a similar thing for Q, except that we define it as $Q \oplus Q^!$ on $\mathcal{E}^\vee \oplus \mathcal{E}^![-1]^\vee$ and 0 on $\mathcal{L}^\vee[-1]$. To define $\{I, \_\}$, we first let I denote the coset of the following element of $\mathcal{L}(U)[1]^\vee \otimes \mathcal{E}^\vee \otimes \mathcal{E}^![-1]^\vee$ in $\widehat{Sym}^\bullet$:

$$X \otimes \phi \otimes \psi \mapsto \langle \phi, X\psi \rangle.$$

To define $\{I, \_\}$ takes a bit of effort. The quickest way to define it is by defining it on elements ϕ of $T^*[-1]\mathcal{E}^\vee$ by

$$\{I, \phi\}(X, \psi) = -\phi([X, \psi])$$

when $X \in \mathcal{L}$ and $\psi \in T^*[-1]\mathcal{E}$. We extend $\{I, \_\}$ to the rest of $\mathrm{Obs}^{cl}(U)$ by demanding that it be zero on elements of $\mathcal{L}[1]^\vee$ and a degree 1 derivation. Diagrammatically, $\{I, \_\}$ is represented by the following picture: DRAW PICTURE

Now that we've described the classical observables, we should move on to the quantum observables. This is the point where I should say that I was lying when I said that the quantum theory is described by the interaction I. We want to deform the differential on $\mathrm{Obs}^{cl}(U)$ by a term $\hbar\Delta$, where Δ is a BV Laplacian. This is what works in the finite-dimensional version of the BV story. However, the Δ as we would want to define it requires pairing distributions with distributions, a big no-no. Our next best solution is to have a family Δ_t of BV Laplacians parametrized by $\mathbb{R} > 0$. The first thing we want to do is define a slight modification of heat kernel which is more useful for our purposes.

Definition 3.3. The **BV heat kernel** $K_t \in T^*[-1]\mathcal{E} \otimes T^*[-1]\mathcal{E}$ is characterized by the property

$$-1 \otimes \langle \_, \_ \rangle (K_t \otimes e) = \exp(-tH)e.$$

In other words,

$$-K_t(x, \langle y), \_\rangle = k_t(x, y),$$

where the notation hopefully explains itself. It should be noted that the heat kernel is a degree one object. Now, we can define

Definition 3.4. (1) The scale t **BV Laplacian** is the order-two operator $-\partial_{K_t}$; in other words, given an element ϕ of $\mathrm{Obs}^{cl}(U)$, we sum (with appropriate signs) over possible ways of putting K_t in two of the "slots" of ϕ.

(2) The **scale t Poisson bracket** $\{\_,\_\}_t$ is defined by

$$\{J, J'\}_t = \Delta_t(JJ') - \Delta_t(J)J' - (-1)^{|J|} J\Delta_t(J').$$

Now, I won't get into the details, but there is a very non-canonical procedure for replacing I with a family of interactions $I[t] \in \mathrm{Obs}^{cl}(U)[[\hbar]]$, one for each t, such that there is a well-defined $t \to 0$ limit modulo $\hbar$ and $I[t \to 0] = I \pmod{\hbar}$. With all of these objects in place, we can define the scale t quantum observables:

Definition 3.5. The scale t quantum observables have the same underlying graded vector space as $\mathrm{Obs}^{cl}(U)$, but with the "pre-"differential

$$Q + d + \{I[t], \_\}_t + \hbar\Delta_t.$$

The important question is whether this differential squares to zero; in fact, it turns out that the $I[t]$s are related in such a way that if the scale t pre-differential squares to zero, then the scale t' differential does as well for any other t'. The failure of this differential to square to zero is what we call the **obstruction** to $\mathcal{L}$-equivariant quantization of $\mathcal{E}$.

3.3. Obstruction Theory.

Let us examine the obstruction more closely. Our main tool is the following theorem

Theorem 3.1 (Gwilliam)**.**

(1) *The obstruction to the $\mathcal{L}$-equivariant quantization of the cotangent theory to $\mathcal{E}$ is given by a well-defined cohomology class $\mathcal{O}(U) \in H^\bullet(\widehat{Sym}(\mathcal{L}[1]^\vee))$.*

(2) *Under conditions which are satisfied if in our key example D is self-adjoint with respect to some Hermitian metric on V and M is compact, then $\mathcal{O}(M)$ is given by the trace of the action of $H^\bullet(\mathcal{L}(M))$ on the determinant of $H^*(\mathcal{E}(M))$. Here we mean the graded determinant: if V is a Z-graded vector space,*

$$\det(V) = \bigotimes_i (\Lambda^{\dim V_i} V_i)^{(-1)^i},$$

with W^{-1} defined as $W^\vee$.

We want to say more about how the obstruction class. To this end, we need the following

Definition 3.6. The **propagator from scale t to scale t'** is a degree 0 section of $T^*[-1]\mathcal{E} \otimes T^*[-1]\mathcal{E}$ given by

$$P(t, t') = \int_{s=t}^{t'} (Q^{GF} \otimes 1) K_s ds.$$

Now, we can describe how to compute the obstruction $\mathcal{O}(U)$. We let

Definition 3.7. The **tree-level, scale t interaction** is the element of $\mathrm{Obs}^q(U)[t]$ given by taking a sum over all connected tree-graphs with trivalent vertices as described above. The internal edges can only be composed of $T^*[-1]\mathcal{E}$ half-edges. To each graph we associate the following element of $\mathrm{Obs}^q(U)[t]$: DRAW IT

Notice that for simple combinatorial reasons, all of the trees contributing to I_{tr} have only two external $T^*[-1]\mathcal{E}$ edges. Thus, $\Delta_t I_{tr}$ belongs to $\widehat{Sym}(\mathcal{L}[1]^\vee)$. More important, we have

Lemma 3.2. *A representative of the obstruction class is given by $\Delta_t I_{tr}[t]$. The cohomology class of this obstruction is independent of t.*

4. McKean-Singer

Now we are ready to use this machinery to derive the McKean-Singer formula. Recall from our key example that we want to take $\mathcal{E} = \mathcal{V}^+ \xrightarrow{D^+} \mathcal{V}^-$, $\mathcal{L} = \Omega^\bullet$, and Q^{GF} to be the negative part of the Dirac operator. We claim that Theorem 3.1 implies the McKean-Singer formula. To see this, let us first work out what the second part of the theorem tells us: in our case $H^\bullet(\mathcal{E})$ has $\ker(D^+)$ in degree 0 and $\operatorname{coker}(D^+)$ in degree 1, so that $\det(H^\bullet(\mathcal{E})) = \Lambda^{\dim \ker(D^+)} \ker(D^+) \otimes (\Lambda^{\dim \operatorname{coker}(D^+)} \operatorname{coker}(D^+))^\vee$. Now, $H^\bullet(\mathcal{L})$ is just the de Rham cohomology of M. In particular $\lambda \in H^0(\mathcal{L})$ acts on $\det(H^\bullet(E))$ by $\lambda \operatorname{ind}(D)$. So, we have one side of the equality. On the other hand, we consider the Feynman diagrams appearing in $I_{tr}[t]$: a graph with n vertices corresponds to the element

$$\lambda^{\otimes n} \otimes \phi \otimes \psi = \lambda^n \langle \phi, (Q^{GF} \int_0^t e^{-sH} ds)^{n-1} \psi \rangle.$$

Since $Q^{GF} e^{-tH}$ lowers cohomological degree by 1, all terms with $n > 2$ are just zero. And the term for $n = 2$ does not vanish, but when we take the BV Laplacian to it, it will vanish. This is because $\langle \phi, (Q^{GF} e^{-tH})^{n-1} \psi \rangle$ is non-zero only when ϕ and ψ sit in the same degree. On the other hand, $K_t \in \mathcal{E}^0 \otimes \mathcal{E}^1 \oplus \mathcal{E}^1 \otimes \mathcal{E}^0$, so when we take the BV Laplacian, we do not get a non-zero contribution. Thus, the only contribution to the obstruction comes from the following diagram: DRAW IT

This diagram gives a contribution computed in the following way: at each point x of M, $k_t(x,x)$ is an element of $V_x^+ \otimes V_x^{+\vee} \oplus V_x^- \otimes V_x^{-\vee}$. We pair these via $\langle , \rangle$, which means that when we pair V_x^+ with $V_x^{+!}$ we get no sign, but we get a minus sign when we pair the degree 1 V^- part with the degree 0 $V^{-!}$ part. Then we integrate over M. What we have shown is that

$$\Delta_t I_{tr}[t](\lambda) = \lambda \int_{x \in M} \operatorname{Str} k_t(x,x) |dx|.$$

This completes the proof.

References

[G] Gwilliam, Owen. Factorization Algebras and Free Field Theories. PhD Thesis. 2012.